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# MEET THE LANGLANDS PROGRAM, THE WORLD’S BIGGEST MATHS PROJECT

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A view of a bridge in Zeeland, the Netherlands, April 22, 2021. The Langlands Program can be likened to building bridges across mathematical cultures with different objects and languages. | Photo Credit: Hilbert Simonse/Unsplash

“I am now elderly and have turned my efforts to other things – just a reading, in the original – of the very long (four volume) Turkish novel, *Bir Ada Hikayesi*,” mathematician Robert Langlands, now 87, told this author over an email.

Five years ago, in 2018, Dr. Langlands was awarded the Abel Prize, one of the highest honours for mathematicians, for “his visionary program connecting representation theory to number theory”.

This program was set in motion in 1967 when Dr. Langlands, then 30 and at Princeton University, wrote a 17-page letter to the French mathematician André Weil with a series of tentative ideas.

Even Wikipedia, which often excels at simplifying intricate ideas, admits that the Langlands Program consists of “very complicated theoretical abstractions, which can be difficult even for specialist mathematicians to grasp”.

At the heart of the Program is an attempt to find connections between two far-flung areas of mathematics: number theory and harmonic analysis.

Number theory is the arithmetic study of numbers and the relationships between them. A famous example of such a relationship is the Pythagoras theorem: $a^2 + b^2 = c^2$.

Harmonic analysis is interested in the study of periodic phenomena. Unlike number theorists, who deal with discrete arithmetics (like integers), harmonic analysts deal with mathematical objects more continuous in nature (like waves).

In 1824, Norwegian mathematician Niels Henrik Abel proved that it was impossible to have a general formula to find the roots of polynomial equations whose highest power is greater than 4 (e.g., $x^5 + 2x^4 – 5x^3 – 9x^2 = 0$).

An example of a general formula is the quadratic formula used to solve quadratic equations.

Around the same time, unaware of Abel's work, French mathematician Évariste Galois arrived at the same conclusion – and went a step ahead. In 1832, he suggested that instead of trying to find the precise roots of such polynomial equations, mathematicians could focus on symmetries between roots for an alternate route.

Consider the polynomial equation $x^2 – 2 = 0$. The two roots of x in this equation are 2 and -2. Now, consider a different polynomial involving one of these roots (say, 2): $2^2 + 2 = 2 + 2$.

This equation – of the form $^2$ +  = 2 + , where  = 2 – holds true for the other root as well: $(-2)^2$ + (-2) = 2 + (-2) = 2 - 2.

So the two roots of the polynomial $x^2 – 2 = 0$ are symmetric. And a Galois group is a collection of symmetries of the roots of a polynomial equation.

The Langlands Program seeks to connect every Galois group with automorphic functions, allowing mathematicians to investigate polynomial equations using tools from calculus, and build a bridge from harmonic analysis to number theory.

Alex Kontorovich, a distinguished professor of mathematics at Rutgers University, has used the following example to illustrate the role of automorphic functions. Let's start with functions of a variable x such that:

$f_1(x) = x$

$f_2(x) = x-x^3/3!$ (The '!' sign means a factorial, so 3! is 3 x 2 x 1)

$f_3(x) = x-x^3/3!+x^5/5!$

$f_4(x) = x-x^3/3!+x^5/5!-x^7/7!$

For each term, the coefficient is the factorial for the same number as the power to which x is raised in that term ($x^{\mathbf{3}}/\mathbf{3}!$, $x^{\mathbf{5}}/\mathbf{5}!$, etc.). To add a new term to the function, we need to follow the alternating pattern of addition and subtraction, raise x to the power of the next odd number, and divide the term by the same number's factorial. So this is how $f_{10}(x)$ would look:

$f_{10}(x)=x-x^3/3!+x^5/5!-x^7/7!+x^9/9!-x^{11}/11!+x^{13}/13!-x^{15}/15!+x^{17}/17!-x^{19}/19!$

Let's plot this function on a graph, with x on the x-axis and f(x) on the y-axis:

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Note that while the function tends to infinity at -10 and 10, in the middle it appears to have the same periodicity as the sine function from trigonometry. That is, according to Prof. Kontorovich, the function f(x) is another way of writing the sine function.

If we note all terms of f(x) until infinity, we would have what mathematicians call an infinite series:

$f(x) = x-x^3/3!+x^5/5!-x^7/7!+x^9/9!-x^{11}/11!+ x^{13}/13!-x^{15}/15!+x^{17}/17!-x^{19}/19!...+...-...$

Since we know the function above to be a different form of the sine function, we can use the properties of the sine function to rewrite it in a simpler form.

For example, the sine function itself can be represented on a circle. If you have a piece of string shaped like the sine wave, you can also bend it to shape it like a circle. If a bead on the string goes from the baseline to the crest of the wave, then down all the way to the trough, and finally returns to the baseline, it would be like travelling from a point on top of the circle to the bottom and back – which is 360º degrees or 2 radians.

So the sine wave can be said to repeat itself after 2 radians, and we can write f(x) thus:

$f(x) = f(x + 2)$

This function is said to have a translational symmetry: despite having been shifted by a factor of 2, the function looks the same. That f(x) has translational symmetry is a "spectacular miracle", according to Prof. Kontorovich.

Such functions that turn back into themselves when the variables are changed by some process are called automorphic functions. The sine function is a simple example.

The Langlands Program is an effort to connect Galois groups to these functions.

In 1994, Andrew Wiles and Richard Taylor applied Langlands' conjectures to prove Fermat's last theorem. This proof had eluded mathematicians for more than three centuries.

The Program has also helped mathematicians create new automorphic functions from preexisting ones. Such possibilities, they understand, could be crucial to prove the Ramanujan conjectures, many of which remain unsolved.

For example, consider the following automorphic function g(x):

$g(x) = a + a_1x + a_2x^2 + a_3x^3 \ldots$

One conjecture of the Program is called functoriality. It posits that we can raise the coefficients of g(x) – a, $a_1$, $a_2$, etc. – to any integer to create a different automorphic function. That is, the following function, where the coefficient has been raised to an integer k, should also be automorphic:

$g_k(x) = a^k + a_1{}^kx + a_2{}^kx^2 + a_3{}^kx^3\ldots$

Since Dr. Langlands' letter to Dr. Weil, the Program has also evolved into its own field of mathematics. One offshoot – called Geometric Langlands – investigates connections between algebraic geometry and representation theory. Mathematicians have even conjectured connections between Geometric Langlands and physics.

Earlier this year, for example, mathematicians David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh found signs of electromagnetism in number theory. In their paper, they recast two different mathematical objects – periods and *L*-functions – into geometric objects that physicists use to study electromagnetic waves.

As such, the Langlands Program is a mathematical exercise in translation – in building bridges across mathematical cultures with different objects and languages.

*The author thanks Vivek Tewary (Krea University) and Mohan R. (Azim Premji University) for their inputs. Sayantan Datta are a queer-trans freelance science writer, communicator and journalist. They are currently a faculty member at Krea University and tweet at @queersprings.*

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